

On a Brownian motion with a hard membrane

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Abstract

Local perturbations of a Brownian motion are considered. As a limit we obtain a non-Markov process that behaves as a reflected Brownian motion on the positive half line until its local time at zero reaches some exponential level, then changes a sign and behaves as a reflected Brownian motion on the negative half line until some stopping time, etc.

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1 Introduction

Consider a sequence of SDEs

$$dX_\varepsilon(t) = a_\varepsilon(X_\varepsilon(t))dt + dw(t), \quad t \geq 0, X_\varepsilon(0) = x, \quad (1)$$

where w is a Wiener process.

We assume that a_ε is an integrable function; this ensures existence and uniqueness of a weak solution to this SDE [1], Theorem 4.53.

We also will suppose that the support of a_ε is contained in $[-\varepsilon, \varepsilon]$; X_ε will be interpreted as a local perturbation of a Brownian motion.

Condition $\text{supp}(a_\varepsilon) \subset [-\varepsilon, \varepsilon]$ ensures the weak relative compactness in the space of continuous functions of $\{X_{\varepsilon_n}\}$ for any sequence $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. The aim of this paper is to discuss possible limits of $\{X_\varepsilon\}$ as $\varepsilon \rightarrow 0+$.

If $a_\varepsilon(x) = \varepsilon^{-1}a(\varepsilon^{-1}x)$, then a_ε converges in the generalized sense to $\alpha\delta$, where $\alpha = \int_{\mathbb{R}} a(x)dx$ and δ is the Dirac delta function at 0. In this case [3, 5] we have convergence in distribution in the space of continuous functions

$$X_\varepsilon \Rightarrow w_\gamma,$$

where $\gamma = \tanh \alpha$, w_γ is a skew Brownian motion, i.e., a continuous Markov process with transition density

$$p_t(x, y) = \varphi_t(x - y) + \gamma \text{sgn}(y) \varphi_t(|x| + |y|), \quad x, y \in \mathbb{R},$$

$\varphi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ is the density of the normal distribution $N(0, t)$.

If $\gamma = 1$ (or $\gamma = -1$), then w_γ is a Brownian motion with reflection into the positive (or negative) half-line.

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Assume that $\text{sgn}(x)a_\varepsilon(x) \geq 0$. Then the drift term pushes up when X_ε is on the positive half line and pushes down when X_ε is on the negative half line. If the limit of sequence $\mathbf{1}_{x \geq 0}a_\varepsilon(x)$ as $\varepsilon \rightarrow 0+$ is “greater than” delta function, then any limit of X_ε cannot cross through zero and consequently the limit will be a reflection Brownian motion.

Note that the skew Brownian with $|\gamma| < 1$ has both positive and negative excursions in any neighborhood of hitting 0 with probability 1. Reflected Brownian motion ($\gamma = 1$) does not cross zero if it starts from $x \geq 0$; otherwise if $x < 0$, then it crosses 0 immediately after the hitting. We find a situation when a limit of $\{X_\varepsilon\}$ is an intermediate regime between a reflecting case and a skew Brownian motion. The limit process will be a reflection Brownian motion in some half line until its local time reaches an exponential random variable. Then it behaves as Brownian motion with reflection into another half line until its local time reaches another independent exponential random variable, etc. We call such process a Brownian motion with a hard membrane. The corresponding definitions are given in §2. We prove the general convergence result in §3. As an example we discuss the case $a_\varepsilon(x) = L_\varepsilon \varepsilon^{-1} a(\varepsilon^{-1}x)$, $\text{supp}(a) \subset [-1, 1]$ in §4.

2 Definitions. Reflecting Brownian motion. Brownian motion with a hard membrane.

Recall the definition and properties of the Skorokhod reflecting problem, see for ex. [4].

Definition 1. Let $f \in C([0, T])$, $f(0) \geq 0$. A pair of continuous functions g and l is called to be a solution of the Skorokhod problem for f if

- S1.** $g(t) \geq 0$, $t \in [0, T]$;
- S2.** $g(t) = f(t) + l(t)$, $t \in [0, T]$;
- S3.** $l(0) = 0$, l is non-decreasing;
- S4.** $\int_0^T \mathbf{1}_{g(s) > 0} dl(s) = 0$.

It is well known that there exists a unique solution to the Skorokhod problem and the solution is given by the formula

$$l(t) = - \min_{s \in [0, t]} (f(s) \wedge 0) = \max_{s \in [0, t]} (-f(s) \vee 0), \quad (2)$$

$$g(t) = f(t) + l(t) = f(t) - \min_{s \in [0, t]} (f(s) \wedge 0). \quad (3)$$

We will say that g is a reflection process into a positive half line and denote it by $f^{refl, +}$.

Remark 1. If $f(0) < 0$, then we set by definition $f^{refl, +}(t) := f(t)$ until the instant ζ_0 of hitting 0, and $f^{refl, +}(t) := f(t) - \min_{s \in [\zeta_0, t]} f(s)$ for $t \geq \zeta_0$.

The reflection problem with reflection into the negative half line is constructed similarly, $g(t) := f(t) - l(t)$, where l is also non-decreasing. In this case denote g by $f^{refl, -}$.

Let $w(t)$, $t \geq 0$ be a Brownian motion started at x , $w^{refl, +}$ and $w^{refl, -}$ be reflected Brownian motions with reflection at 0 into the positive and negative half lines, correspondingly. It is known that the process $l(t)$ is the two-sided local time of $w^{refl, \pm}$ at 0 defined by $\lim_{\varepsilon \rightarrow 0+} (2\varepsilon)^{-1} \int_0^t \mathbf{1}_{[-\varepsilon, \varepsilon]}(w^{refl, \pm}(s)) ds$.

Consider two sequences of exponential random variables $\{\xi_k^+\}$ and $\{\xi_k^-\}$ with parameters α^\pm , respectively. Suppose that all random variables $\{\xi_k^\pm\}$ and the Brownian motion $w(t)$, $t \geq 0$ are mutually independent.

Assume that $w(0) = x > 0$. The Brownian motion with a hard membrane and parameters of permeability α^\pm is constructed in the following way.

$$w^{hard}(t) := w^{refl,+}(t) \text{ if } l(t) \leq \xi_1^+.$$

At instant $\zeta_1^+ := \inf\{t \geq 0 \mid l(t) \geq \xi_1^+\}$ the process w^{hard} changes orientation. It reflects into negative half line until the moment when its local time after ζ_1^+ reaches the level ξ_1^- :

$$w^{hard}(t) := w(t) - w(\zeta_1^+) - (l(t) - l(\zeta_1^+)) = w(t) - w(\zeta_1^+) - \max_{s \in [\zeta_1^+, t]} (w(s) - w(\zeta_1^+))$$

for $t \leq \inf\{t \geq \zeta_1^+ : (l(t) - l(\zeta_1^+)) \geq \xi_1^-\} = \inf\{t \geq \zeta_1^+ : \max_{s \in [\zeta_1^+, t]} (w(s) - w(\zeta_1^+)) \geq \xi_1^-\}$.

Denote $\zeta_1^- := \inf\{t \geq \zeta_1^+ : \max_{s \in [\zeta_1^+, t]} (w(s) - w(\zeta_1^+)) \geq \xi_1^-\}$. At the moment ζ_1^- changes its orientation again and reflects into the positive half line, and so on. The formal equation for w^{hard} is the following

$$w^{hard}(t) = w(t) + \int_0^t \left(\mathbb{1}_{l(s) \in \cup [\xi_1^+ + \xi_1^- + \dots + \xi_k^+ + \xi_k^-, \xi_1^+ + \xi_1^- + \dots + \xi_k^+ + \xi_k^- + \xi_{k+1}^+]} - \right. \\ \left. - \mathbb{1}_{l(s) \in \cup [\xi_1^+ + \xi_1^- + \dots + \xi_k^+, \xi_1^+ + \xi_1^- + \dots + \xi_k^+ + \xi_k^-]} \right) dl(s).$$

If $x = w(0) < 0$, then process $w^{hard}(t), t \geq 0$ is constructed similarly.

In case $w(0) = 0$, we have to “attach” the initial direction of reflection at zero. Denote the direction of reflection of w^{hard} at time t by $\text{sgn } w^{hard}(t) \in \{-1, 1\}$. Note that $\text{sgn } w^{hard}(t) = 1$ if $w^{hard}(t) > 0$ and $\text{sgn } w^{hard}(t) = -1$ if $w^{hard}(t) < 0$; $\text{sgn } w^{hard}(t)$ may change a sign only at instants $\xi_1^+ + \xi_1^- + \dots + \xi_k^+ + \xi_k^-$ or $\xi_1^+ + \xi_1^- + \dots + \xi_k^+$. We will always select a cadlag modification for $\text{sgn } w^{hard}(t), t \geq 0$. It can be seen that the pair $(w^{hard}(t), \text{sgn } w^{hard}(t))$ is a strong Markov process on $\mathbb{R} \times \{-1, 1\}$. We can informally consider $w^{hard}(t), t \geq 0$ as the strong Markov process on the set $(-\infty, 0-] \cup [0+, \infty)$, where $0\pm$ means direction of the reflection at when $w(0) = 0$.

3 General conditions of convergence

We assume that $\text{supp } a_\varepsilon \subset [-\varepsilon, \varepsilon]$, $\text{sgn}(x)a_\varepsilon(x) \geq 0$, $a_\varepsilon \in L_1(\mathbb{R})$.

Let us suppose for simplicity that $X_\varepsilon(0) = x > 0$. Introduce a sequence of stopping times

$$\sigma_0^{(\varepsilon)} := 0; \\ \tau_{n+1}^{(\varepsilon)} := \inf\{t \geq \sigma_{n+1}^{(\varepsilon)} : X_\varepsilon(t) = \varepsilon\}, \quad n \geq 0; \\ \sigma_n^{(\varepsilon)} := \inf\{t \geq \tau_n^{(\varepsilon)} : X_\varepsilon(t) = 2\varepsilon\}, \quad n \geq 1.$$

Remark 2. For any $n \geq 1$ moments $\sigma_n^{(\varepsilon)}, \tau_n^{(\varepsilon)}$ are finite a.s., and

$$\lim_{n \rightarrow \infty} \sigma_n^{(\varepsilon)} = \lim_{n \rightarrow \infty} \tau_n^{(\varepsilon)} = \infty \text{ a.s.}$$

Set

$$A_\varepsilon(t) := \int_0^t \mathbb{1}_{X_\varepsilon(s) \in \cup_k [\sigma_k^{(\varepsilon)}, \tau_k^{(\varepsilon)}]} ds, \\ A_\varepsilon^{(-1)}(t) := \inf\{s \geq 0 : A_\varepsilon(s) \geq t\},$$

$$\bar{X}_\varepsilon(t) := X_\varepsilon(A_\varepsilon^{(-1)}(t)), \quad \bar{w}_\varepsilon(t) := \int_0^{A_\varepsilon^{(-1)}(t)} \mathbb{1}_{X_\varepsilon(s) \in \cup_k [\sigma_k^{(\varepsilon)}, \tau_k^{(\varepsilon)}]} dw(s).$$

Observe that $\bar{w}_\varepsilon(t), t \geq 0$ is a Wiener process.

Let $\bar{n}_\varepsilon(t)$ be the number of hitting ε by a process \bar{X}_ε until time $t \geq 0$.

Set

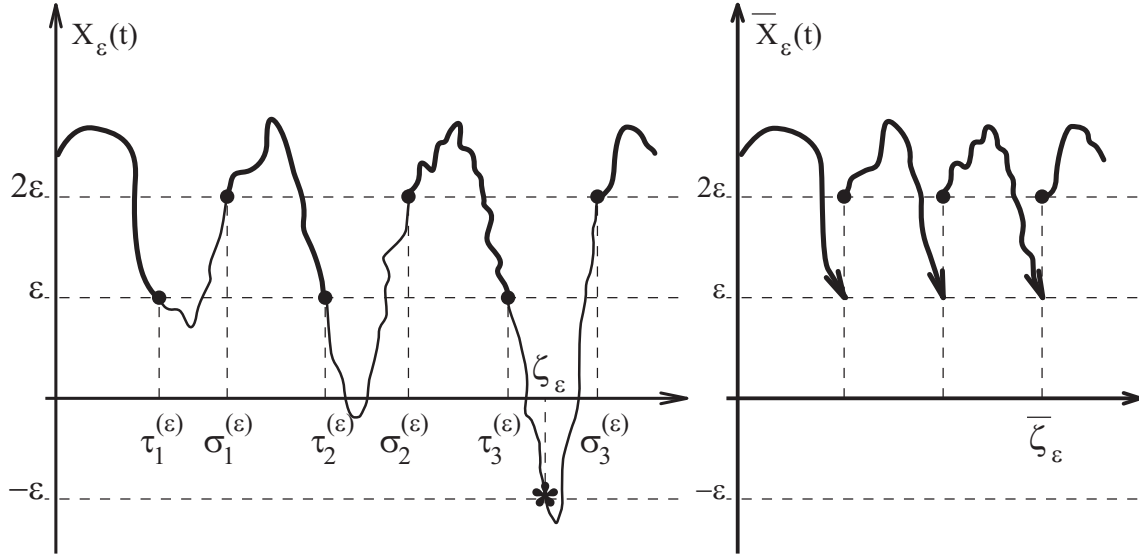
$$\bar{G}_\varepsilon = \inf \left\{ k \geq 0 : \sigma_k^{(\varepsilon)} > \inf \{ t \geq \tau_k^{(\varepsilon)} : X_\varepsilon(t) = -\varepsilon \} \right\}.$$

Define

$$\zeta_\varepsilon := \inf \{ t \geq 0 : X_\varepsilon(t) = -\varepsilon \},$$

$$\bar{\zeta}_\varepsilon = A_\varepsilon(\zeta_\varepsilon) = \int_0^{\zeta_\varepsilon} \mathbb{1}_{X_\varepsilon(s) \in \cup_k [\sigma_k^{(\varepsilon)}, \tau_k^{(\varepsilon)}]} ds = \sum_{k=0}^{\bar{G}_\varepsilon-1} (\tau_{k+1}^{(\varepsilon)} - \sigma_k^{(\varepsilon)}).$$

Figure 1:



Informally, the behavior of \bar{X}_ε is the following. It moves as a Brownian motion until it hits ε . Next \bar{X}_ε immediately jumps to 2ε and moves as a Brownian motion (a shift of w starting at stopping time) again until the second hitting ε , then jumps to 2ε , etc.

Set

$$p_\varepsilon^+ = P(\inf \{ t \geq 0 : X_\varepsilon(t) = -\varepsilon \} < \inf \{ t \geq 0 : X_\varepsilon(t) = 2\varepsilon \} \mid X_\varepsilon(0) = \varepsilon); \quad (4)$$

p_ε^+ is the probability for the process X_ε to reach $-\varepsilon$ before 2ε given $X_\varepsilon(0) = \varepsilon$.

Remark 3. We will always assume that $\{w(t)\}$ is a Wiener process started from a fixed point x .

Lemma 1. Assume that $X_\varepsilon(0) = x$, where $x > \varepsilon$.

Set

$$\tilde{X}_\varepsilon(t) = w(t) + \varepsilon \tilde{n}_\varepsilon(t),$$

where

$$\tilde{n}_\varepsilon(t) := \left[-\varepsilon^{-1} \min_{s \in [0, t]} ((w(s) - \varepsilon) \wedge 0) \right]$$

and $[\cdot]$ is the integer part of a number.

Let \tilde{G}_ε^+ be a geometrical random variable with parameter p_ε^+

$$P(\tilde{G}_\varepsilon^+ = n) = (1 - p_\varepsilon^+)^{n-1} p_\varepsilon^+, \quad n \geq 1;$$

and \tilde{G}_ε^+ is independent of a Wiener process $w(t), t \geq 0$.

Denote $\tilde{\zeta}_\varepsilon := \inf\{t \geq 0 : \tilde{n}_\varepsilon(t) \geq \tilde{G}_\varepsilon^+\}$.

Then the distributions in $D([0, \infty))^2 \times C([0, \infty)) \times \mathbb{R}$ of quadruples $(\bar{X}_\varepsilon(\cdot), \bar{n}_\varepsilon(\cdot), \bar{w}_\varepsilon(\cdot), \bar{\zeta}_\varepsilon)$ and $(\tilde{X}_\varepsilon(\cdot), \tilde{n}_\varepsilon(\cdot), w(\cdot), \tilde{\zeta}_\varepsilon)$ are equal.

The proof follows from the strong Markov property of the Wiener process.

Theorem 1. Let $X_\varepsilon(0) = x > 0$. Assume that $\varepsilon p_\varepsilon^+ \rightarrow \alpha^+$ as $\varepsilon \rightarrow 0 +$. Let $Exp(\alpha^+)$ be exponential random variable that is independent of w . Then $(\tilde{X}_\varepsilon(\cdot), \varepsilon \tilde{n}_\varepsilon(\cdot), w(\cdot), \tilde{\zeta}_\varepsilon)$ converges weakly to $(w^{refl,+}(\cdot), l^+(\cdot), w(\cdot), \zeta)$ in $D([0, \infty))^2 \times C([0, \infty)) \times \mathbb{R}$, where $(w^{refl,+}, l^+)$ is a solution of the Skorokhod problem with reflection into the positive half line (see §2), $\zeta = \inf\{t \geq 0 : l^+(t) \geq Exp(\alpha^+)\}$.

Proof. It follows from Lemma 1 and (2), (3) that $(\tilde{X}_\varepsilon(\cdot), \varepsilon \tilde{n}_\varepsilon(\cdot), w)$ converges uniformly on compact sets to $(w^{refl,+}(\cdot), l^+(\cdot), w(\cdot))$.

It can be easily seen from the definition of l^+ that for any $c > 0$ the instant $\zeta_c = \inf\{t \geq 0 : l^+(t) \geq c\}$ is a point of increase of l^+ with probability 1, that is

$$P(\forall \delta > 0 : l^+(\zeta_c + \delta) > l^+(\zeta_c)) = 1.$$

Hence, the independence of $Exp(\alpha^+)$ and w yields that the moment $\zeta := \inf\{t \geq 0 : l^+(t) \geq Exp(\alpha^+)\}$ is also a point of increase of l^+ with probability 1.

Since $\varepsilon Geom(p_\varepsilon^+) \Rightarrow Exp(\alpha^+)$ if $\varepsilon p_\varepsilon^+ \rightarrow \alpha^+$, this completes the proof. \square

Corollary 1. Assume that $\varepsilon p_\varepsilon^+ \rightarrow \alpha^+$ as $\varepsilon \rightarrow 0 +$. Then we have the weak convergence

$$(\bar{X}_\varepsilon(\cdot \wedge \bar{\zeta}_\varepsilon), \varepsilon \bar{n}_\varepsilon(\cdot \wedge \bar{\zeta}_\varepsilon), \bar{w}_\varepsilon(\cdot \wedge \bar{\zeta}_\varepsilon), \bar{\zeta}_\varepsilon) \Rightarrow (w^{refl,+}(\cdot \wedge \zeta), l^+(\cdot \wedge \zeta), w(\cdot \wedge \zeta), \zeta)$$

as $\varepsilon \rightarrow 0 +$ in $D([0, \infty))^2 \times C([0, \infty)) \times \mathbb{R}$, where $\zeta = \inf\{t \geq 0 : l^+(t) \geq Exp(\alpha^+)\}$.

Theorem 2. Assume that $\varepsilon p_\varepsilon^+ \rightarrow \alpha^+$ as $\varepsilon \rightarrow 0 +$. Then we have the weak convergence

$$(X_\varepsilon(\cdot \wedge \zeta_\varepsilon), w(\cdot \wedge \zeta_\varepsilon), \zeta_\varepsilon) \Rightarrow (w^{refl,+}(\cdot \wedge \zeta), w(\cdot \wedge \zeta), \zeta),$$

where $\zeta_\varepsilon = \inf\{t \geq 0 : X_\varepsilon(t) = -\varepsilon\}$, $\zeta = \inf\{t \geq 0 : l^+(t) \geq Exp(\alpha^+)\}$.

Proof. Assume that $\zeta_\varepsilon \in [0, T]$. Then by construction of $\bar{\zeta}_\varepsilon, \bar{X}_\varepsilon, \bar{w}_\varepsilon$ we have

$$\begin{aligned} |\bar{\zeta}_\varepsilon - \zeta_\varepsilon| &\leq \int_0^T \mathbb{1}_{|X_\varepsilon(s)| < 2\varepsilon} ds; \\ \sup_{t \in [0, T]} |\bar{w}_\varepsilon(t \wedge \bar{\zeta}_\varepsilon) - w(t \wedge \zeta_\varepsilon)| &\leq 2\omega_{w, [0, T]} \left(\int_0^T \mathbb{1}_{|X_\varepsilon(s)| < 2\varepsilon} ds \right); \\ \sup_{t \in [0, T]} |\bar{X}_\varepsilon(t \wedge \bar{\zeta}_\varepsilon) - X_\varepsilon(t \wedge \zeta_\varepsilon)| &\leq 3\varepsilon + \omega_{X_\varepsilon, [0, T]} \left(\int_0^{\zeta_\varepsilon} \mathbb{1}_{|X_\varepsilon(s)| < 2\varepsilon} ds \right) \leq \\ &6\varepsilon + 2\omega_{w, [0, T]} \left(\int_0^T \mathbb{1}_{|X_\varepsilon(s)| < 2\varepsilon} ds \right). \end{aligned}$$

Here $\omega_{f,[0,T]}(\delta) := \sup_{s,t \in [0,T], |s-t| < \delta} |f(s) - f(t)|$ is the modulus of continuity of f .

The proof of the Theorem will follow from Corollary 1 if we show that for any $T > 0$

$$\int_0^T \mathbb{1}_{|X_\varepsilon(s)| < 2\varepsilon} ds \xrightarrow{P} 0, \quad \varepsilon \rightarrow 0+.$$
 (5)

By Ito-Tanaka's formula we have

$$|X_\varepsilon(t)| = |x| + \int_0^t \text{sgn}(X_\varepsilon(s)) a_\varepsilon(X_\varepsilon(s)) ds + B(t) + l_\varepsilon(t),$$

where $B(t) = \int_0^t \text{sgn}(X_\varepsilon(s)) dw(s)$ is a new Brownian motion, and l_ε is the local time of X_ε at 0. Notice that $(|X_\varepsilon(t)|, l_\varepsilon(t))$ is a solution of the Skorokhod reflection problem for the process $|x| + \int_0^t \text{sgn}(X_\varepsilon(s)) a_\varepsilon(X_\varepsilon(s)) ds + B(t)$, $t \geq 0$. Since $\text{sgn}(x) a_\varepsilon(x) \geq 0$, formulas (2), (3) yield that $|X_\varepsilon(t)| \geq B^{refl}(t) := |x| + B(t) - \min_{s \in [0,t]} ((|x| + B(s)) \wedge 0)$. Reflecting Brownian motion spends zero time at 0. This yields (5) and hence completes the proof of the Theorem. \square

Denote

$$p_\varepsilon^- := P\left(\inf\{t \geq 0 : X_\varepsilon(t) = \varepsilon\} < \inf\{t \geq 0 : X_\varepsilon(t) = -2\varepsilon\} \mid X_\varepsilon(0) = -\varepsilon\right).$$

Corollary 2. Assume that

$$\lim_{\varepsilon \rightarrow 0+} \varepsilon p_\varepsilon^\pm = \alpha^\pm \in (0, \infty). \quad (6)$$

Then X_ε converges in distribution to a Brownian motion with a hard membrane and parameters α^\pm .

Remark 4. If $\{\alpha^+ = 0 \text{ and } \alpha^- > 0\}$ or $\{\alpha^+ < \infty \text{ and } \alpha^- = \infty\}$, then $X_\varepsilon \Rightarrow w^{refl,+}$ as $\varepsilon \rightarrow 0+$.

4 Example

In this Section we give sufficient conditions that ensure convergence of a sequence $\{X_\varepsilon\}$ to a Brownian motion with a hard membrane. Here we assume that $a_\varepsilon(x) = L_\varepsilon \varepsilon^{-1} a(\varepsilon^{-1} x)$, $\text{supp } a \subset [-1, 1]$, $\text{sgn}(x) a(x) \geq 0$.

We need to verify condition (6).

It is well known [6, 7, 8] that probabilities in (6) are of the form

$$p_\varepsilon^+ = \frac{s(2\varepsilon) - s(\varepsilon)}{s(2\varepsilon) - s(-\varepsilon)}, \quad p_\varepsilon^- = \frac{s(-2\varepsilon) - s(-\varepsilon)}{s(-2\varepsilon) - s(\varepsilon)}, \quad (7)$$

where $s(x) = s_\varepsilon(x) = \int_0^x \exp(-2 \int_0^y a_\varepsilon(z) dz) dy$ is the scale function of the diffusion X_ε .

Denote $A(x) := \int_0^x a(y) dy$. Then

$$\begin{aligned} s_\varepsilon(x) &= \int_0^x \exp(-2 \int_0^y a_\varepsilon(z) dz) dy = \int_0^x \exp(-2 \int_0^y L_\varepsilon \varepsilon^{-1} a(\varepsilon^{-1} z) dz) dy = \\ &= \int_0^x \exp(-2 \int_0^{\varepsilon^{-1} y} L_\varepsilon a(u) du) dy = \int_0^x \exp(-2 L_\varepsilon A(\varepsilon^{-1} y)) dy = \\ &= \varepsilon \int_0^{\varepsilon^{-1} x} \exp(-2 L_\varepsilon A(y)) dy. \end{aligned}$$

So

$$p_\varepsilon^+ = \frac{\int_1^2 \exp(-2L_\varepsilon A(y)) dy}{\int_{-1}^2 \exp(-2L_\varepsilon A(y)) dy}.$$

Since $a(x) = 0, x > 1$, we have for $x > 0$:

$$A(x) = A(1) = \int_0^1 a(y) dy = \int_0^\infty a(y) dy =: A_+.$$

So

$$\varepsilon^{-1} p_\varepsilon^+ = \frac{\varepsilon^{-1} \exp(-2L_\varepsilon A_+)}{\int_{-1}^2 \exp(-2L_\varepsilon A(y)) dy}. \quad (8)$$

Theorem 3. Assume that

$$2A(x) \sim c_\pm |x|^\lambda, x \rightarrow 0\pm, \quad (9)$$

where $c_\pm > 0, \lambda > 0$.

Then

$$\lim_{\varepsilon \rightarrow 0+} \varepsilon^{-1} p_\varepsilon^+ = \alpha_+ > 0 \quad (10)$$

is equivalent to the following condition

$$L_\varepsilon = (2A_+)^{-1} \left(\ln(\varepsilon^{-1}) + \lambda^{-1} \ln \ln(\varepsilon^{-1}) - \right. \\ \left. - (\ln(\alpha_+) + \ln(\Gamma(1 + \lambda^{-1}))) + \lambda^{-1} \ln(2A_+) + \ln(c_+^{-1/\lambda} + c_-^{-1/\lambda}) \right) + o(1), \quad \varepsilon \rightarrow 0+. \quad (11)$$

Proof. It can be seen that $p_\varepsilon^+ \rightarrow 0$ only if $L_\varepsilon \rightarrow +\infty$. The function A attains a minimum at 0. So, for any $\delta > 0$ formula (9) yields ([9], Ch.2, Lemma 1.3)

$$\int_{-1}^2 \exp(-2L_\varepsilon A(y)) dy \sim \int_{-\delta}^\delta \exp(-2L_\varepsilon A(y)) dy \sim \quad (12) \\ \sim \int_0^\delta \exp(-L_\varepsilon c_- y^\lambda) dy + \int_0^\delta \exp(-L_\varepsilon c_+ y^\lambda) dy \sim \Gamma(1 + \lambda^{-1}) L_\varepsilon^{-1/\lambda} (c_-^{-1/\lambda} + c_+^{-1/\lambda})$$

as $\varepsilon \rightarrow 0+$.

Assume that (10) is true. Then it follows from (8) and (12) that

$$-\ln \alpha_+ + o(1) = -\ln \varepsilon - 2A_+ L_\varepsilon + \lambda^{-1} \ln L_\varepsilon - \ln \left(\Gamma(1 + \lambda^{-1}) (c_-^{-1/\lambda} + c_+^{-1/\lambda}) \right) \quad (13)$$

as $\varepsilon \rightarrow 0+$.

Hence $-2A_+ L_\varepsilon \sim \ln \varepsilon$. So $L_\varepsilon = (2A_+)^{-1} \ln(\varepsilon^{-1}) (1 + f(\varepsilon))$, where $f(\varepsilon) \rightarrow 0, \varepsilon \rightarrow 0$. Substituting this into (13), we obtain after cancellations

$$-f(\varepsilon) \ln(\varepsilon^{-1}) + \lambda^{-1} (\ln \ln(\varepsilon^{-1}) - \ln(2A_+)) - \\ - \ln \left(\Gamma(1 + \lambda^{-1}) (c_-^{-1/\lambda} + c_+^{-1/\lambda}) \right) = \ln \alpha_+ + o(1).$$

Hence

$$f(\varepsilon) = \frac{\lambda^{-1} (\ln \ln(\varepsilon^{-1}) - \ln(2A_+)) - \ln \left(\Gamma(1 + \lambda^{-1}) (c_-^{-1/\lambda} + c_+^{-1/\lambda}) \right) - \ln \alpha_+ + o(1)}{\ln(\varepsilon^{-1})}$$

and we get (11).

The proof that (8) follows from (11) is similar. □

If $\int_{-1}^1 a(y)dy = 0$, then $2A_+ = \int_{-\infty}^{\infty} |a(y)|dy =: \|a\|_1$. Next result follows from Theorem 3 and Corollary 1.

Theorem 4. *Let $\int_{-1}^1 a(y)dy = 0$ and $2A(x) \sim c_{\pm}|x|^{\lambda}, x \rightarrow 0\pm$, where $c_{\pm} > 0, \lambda > 0$.*

Assume that

$$L_{\varepsilon} = (\|a\|_1)^{-1} \left(\ln(\varepsilon^{-1}) + \lambda^{-1} \ln \ln(\varepsilon^{-1}) - \left(\ln(\alpha) + \ln(\Gamma(1 + \lambda^{-1})) + \lambda^{-1} \ln(\|a\|_1) + \ln(c_+^{-1/\lambda} + c_-^{-1/\lambda}) \right) \right) + o(1), \quad \varepsilon \rightarrow 0+,$$

where $\alpha > 0$.

Let $X_{\varepsilon}(0) = x \neq 0, \varepsilon > 0$.

Then $\{X_{\varepsilon}\}$ converges in distribution to the Brownian motion with a hard membrane, where $\alpha_+ = \alpha_- = \alpha$.

Remark 5. The limit process has the same parameters α_+ and α_- despite c_{\pm} may be different. It may seem strangely, however this formula can be obtained in the different way that contains “one-sided” arguments. Similarly to the proof of Theorems 2, 3 it can be shown that if

$$L_{\varepsilon} = (2A_+)^{-1} \left(\ln(\varepsilon^{-1}) + \lambda^{-1} \ln \ln(\varepsilon^{-1}) - \left(\ln \beta + \ln(\Gamma(1 + \lambda^{-1})) + \lambda^{-1} \ln(2A_+) \right) \right) + o(1), \quad \varepsilon \rightarrow 0+, \quad (14)$$

then

$$\lim_{\varepsilon \rightarrow 0+} \varepsilon^{-1} P \left(\zeta_0^{(\varepsilon)} < \zeta_{2\varepsilon}^{(\varepsilon)} \mid X_{\varepsilon}(0) = \varepsilon \right) = c_+^{1/\lambda} \beta, \quad (15)$$

where $\zeta_r^{(\varepsilon)} = \inf\{t \geq 0 : X_{\varepsilon}(t) = r\}$. Moreover

$$X_{\varepsilon}(\cdot \wedge \zeta_0^{(\varepsilon)}) \Rightarrow w^{refl,+}(\cdot \wedge \zeta(c_+^{1/\lambda} \beta)), \quad (16)$$

where $\zeta(c_+^{1/\lambda} \beta) = \inf\{t \geq 0 : l^+(t) \geq \text{Exp}(c_+^{1/\lambda} \beta)\}$.

It can be proved (again similarly to the proof of Theorem 3) that

$$\lim_{\varepsilon \rightarrow 0+} P \left(\zeta_{-2\varepsilon}^{(\varepsilon)} < \zeta_{2\varepsilon}^{(\varepsilon)} \mid X_{\varepsilon}(0) = 0 \right) = \frac{c_+^{-1/\lambda}}{c_+^{-1/\lambda} + c_-^{-1/\lambda}}. \quad (17)$$

Hence, if $c_+^{1/\lambda} \frac{c_+^{-1/\lambda}}{c_+^{-1/\lambda} + c_-^{-1/\lambda}} \beta = \alpha$, i.e., $\beta = (c_+^{-1/\lambda} + c_-^{-1/\lambda}) \alpha$, then (15), (16), (17), and memorylessness of exponential distribution imply Theorem 4.

Remark 6. Suppose that some conditions of Theorem 4 are not satisfied. Let $X_{\varepsilon}(0) = x > 0$. It can be seen that in each of the following cases $X_{\varepsilon} \Rightarrow w^{refl,+}$ as $\varepsilon \rightarrow 0+$

- L_{ε} has a form (14) and either $A_+ > A_-$ or $2A(x) \sim c_{\pm}|x|^{\lambda}, x \rightarrow 0\pm$, where $c_{\pm} > 0$ and $0 < \lambda_+ < \lambda_-$.
- $2A(x) \sim c_+ x^{\lambda_+}, x \rightarrow 0+$ and for some $\delta > 0$

$$L_{\varepsilon} \geq (2A_+)^{-1} \left(\ln(\varepsilon^{-1}) + (1 + \delta) \lambda^{-1} \ln \ln(\varepsilon^{-1}) \right) + o(1).$$

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